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On nonlocal symmetries of some shallow water equations

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Abstract

A recent construction of nonlocal symmetries for the Korteweg–de Vries, Camassa–Holm and Hunter–Saxton equations is reviewed, and it is pointed out that—in the Camassa–Holm and Hunter–Saxton case—these symmetries can be considered as (nonlocal) symmetries of integro-differential equations.

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1. Introduction

Three very important shallow water models first studied in [7, 8, 11, 24, 25, 27], the Korteweg–de Vries (KdV), Camassa–Holm (CH) and Hunter–Saxton (HS) equations, present quite different qualitative features (for instance, *all* smooth solutions to KdV exist globally in time [40], but CH and HS admit smooth solutions which develop singularities in finite time [43, 15] *and* smooth solutions which exist for all times [9, 13]) but, on the other hand, they can be considered profitably from unified points of view: In [3, 4] R Beals, D Sattinger and J Szmigielski analysed these equations from the point of view of scattering/inverse scattering; in [26] B Khesin and G Misiołek showed that they exhaust, in a precise sense, the bi-Hamiltonian equations which can be modelled as geodesic flows on (homogeneous spaces related to) the Virasoro group; finally in [37, 38] the present author pointed out that the existence of zero curvature formulations, quadratic pseudopotentials, modified versions, 'Miura transformations', conservation laws, and nonlocal symmetries for the KdV, CH and HS equations follow from the fact that they belong to a same family of equations describing pseudo-spherical surfaces, a concept reviewed in [36–38] and references therein.

This paper has two goals. First, to give a short account of the constructions appearing in [38]; specifically, to show how to obtain nonlocal symmetries for the Korteweg–de Vries (KdV) Camassa–Holm (CH) and Hunter–Saxton (HS) equations using quadratic pseudo-potentials, in the spirit of [23, 31, 32] and the recent [39]. Second, to remark that, in the Hunter–Saxton and Camassa–Holm case, these new symmetries can be understood as (nonlocal) symmetries of integro-differential versions of these two equations, to be recalled below. This last observation appears to be of interest of its own, as it provides further motivation for the geometric study of symmetries in the interesting—and still under-developed—case of integro-differential equations [12, 29, 44].

The following notation will be used throughout, following [33]: independent variables will be denoted by $x^i, i = 1, 2, \ldots, n$, and dependent variables by $u^\alpha, \alpha = 1, 2, \ldots, m$; k-tuples $J = (j_1, \ldots, j_k), 0 \leqslant j_1, j_2, \ldots, j_k \leqslant n$ will denote multi-indices of order $\#J = k$; partial derivatives with respect to x^i will be denoted by subindices; finally, D_i will indicate the total derivative with respect to x^i,

$$D_i = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{m} \sum_{\#J \geqslant 0} u^\alpha_{Ji} \frac{\partial}{\partial u^\alpha_J},$$

in which $u^\alpha_{Ji} = \partial u^\alpha_J / \partial x^i$.

2. On nonlocal symmetries

Nonlocal symmetries of differential equations were first studied rigorously by A Vinogradov and I Krasil'shchik in their seminal paper [42]. A complete account of their research appears in [28, 29]; a short review and an application of ideas from [28, 29] to the KdV, Camassa–Holm and Hunter–Saxton equations is in [38], and an operational version of their theory has been recently advanced in [39]. Other approaches to nonlocal symmetries are considered in [1, 5, 6, 10, 34, 35] and references therein.

Definition 1. *A covering π of a system of partial differential equations $\Xi_a = 0$ is a triplet*

$$(\{\gamma_b : b = 1, \ldots, N\}; \{X_{ib} : b = 1, \ldots, N; i = 1, \ldots, n\}; \{\widetilde{D}_i : i = 1, \ldots, n\}) \qquad (1)$$

of variables γ_b, smooth functions X_{ib} depending on x^i, u^α, γ_b and x^i-derivatives of u^α, and linear operators

$$\widetilde{D}_i = D_i + \sum_{b=1}^{N} X_{ib} \frac{\partial}{\partial \gamma_b}, \qquad (2)$$

such that the equations

$$\widetilde{D}_i(X_{jb}) = \widetilde{D}_j(X_{ib}), \qquad i, j = 1, 2, \ldots, n, \quad b = 1, 2, \ldots, N, \qquad (3)$$

hold whenever $u^\alpha(x^i)$ is a solution to $\Xi_a = 0$.

It will be understood hereafter that the index i runs from 1 to n and that the index b runs from 1 to N, so that instead of (1) one will simply write $\pi = (\gamma_b; X_{ib}; \widetilde{D}_i)$. The variables γ_b are new dependent variables—the 'nonlocal variables' of the theory—and the operators $\widetilde{D}_i$ satisfying equation (3) are new total derivatives which take into account the nonlocal variables γ_b. Note that the operators $\widetilde{D}_i$ satisfy $\widetilde{D}_i(\gamma_b) = X_{ib}$, and that these equations are compatible because (3) holds. Since *on solutions to the system of equations* $\Xi_a = 0$ the total derivatives $\widetilde{D}_i$ become ordinary partial derivatives, the equations

$$\frac{\partial \gamma_b}{\partial x^i} = X_{ib} \qquad (4)$$

should hold for each index b and each index i *whenever $u^\alpha(x^i)$ is a solution to* $\Xi_a = 0$: these compatible equations specify the relations between the dependent variables u^α and the variables γ_b.

The nonlocal version of the *formal linearization* of the system $\Xi_a = 0$ is the matrix

$$\widetilde{\Xi}_* = \left(\sum_L \frac{\partial \Xi_a}{\partial u^\alpha_L}\widetilde{D}_L\right) = \begin{pmatrix} \dfrac{\partial \Xi_1}{\partial u^1} + \cdots + \dfrac{\partial \Xi_1}{\partial u^1_I}\widetilde{D}_I & \dfrac{\partial \Xi_1}{\partial u^2} + \cdots + \dfrac{\partial \Xi_1}{\partial u^2_J}\widetilde{D}_J & \cdots \\ \dfrac{\partial \Xi_2}{\partial u^1} + \cdots + \dfrac{\partial \Xi_2}{\partial u^1_K}\widetilde{D}_K & \cdots & \cdots \\ \vdots & & \end{pmatrix}. \tag{5}$$

Definition 2. *Let $\Xi_a = 0$ be a system of differential equations and let $\pi = (\gamma_b; X_{ib}; \widetilde{D}_i)$ be a covering of $\Xi_a = 0$. An ordered $(m+N)$-tuple of functions (G^α, H_b) depending on x^i, u^α, γ_b and a finite number of x^i-derivatives of u^α, is a nonlocal π-symmetry of $\Xi_a = 0$ if and only if the equations*

$$\widetilde{\Xi}_*(G) = 0, \tag{6}$$

and

$$\widetilde{D}_i(H_b) = \widetilde{D}_\tau(X_{ib}), \tag{7}$$

hold whenever $u^\alpha(x^i)$ is a solution to $\Xi_a = 0$, in which G is the vector $(G^1, G^2, \ldots, G^m)^t$, and the operator $\widetilde{D}_\tau$ appearing in (7) is given by

$$\widetilde{D}_\tau = \sum_{\alpha=1}^{m}\sum_{\#K\geqslant 0}\widetilde{D}_K(G^\alpha)\frac{\partial}{\partial u^\alpha_K} + \sum_{c=1}^{N} H_c\frac{\partial}{\partial \gamma_c}. \tag{8}$$

Note that equation (6) depends only on the vector G and the system $\Xi_a = 0$: following Krasil'shchik and Vinogradov [28, 29, 42] one says that G is the π-*shadow* of the nonlocal π-symmetry (G^α, H_b). Also important is to note that the differential operator $\widetilde{D}_\tau$ defined in (8) is the nonlocal version of the infinite prolongation [33] of the vector field

$$G^\alpha\frac{\partial}{\partial u^\alpha} + H_b\frac{\partial}{\partial \gamma_b},$$

taking into account the fact that the derivatives of the new dependent variables γ_b can be written in terms of the variables x^i, u^α, u^α_J, γ_b due to equation (4).

One usually says 'nonlocal symmetry' instead of 'nonlocal π-symmetry', and assumes that a covering (1) of $\Xi_a = 0$ has been fixed. The fact that this approach to nonlocal symmetries depends essentially on coverings implies that one should perhaps consider nonlocal symmetries as properly generalizing the class of *intrinsic* symmetries studied in [2]. Further comments on this point appear in [39].

Proposition 1. *If (G^α, H_b) is a nonlocal π-symmetry of the system $\Xi_a = 0$, in which the covering π is given by (1), then the vector field*

$$G^\alpha\frac{\partial}{\partial u^\alpha} + H_b\frac{\partial}{\partial \gamma_b} \tag{9}$$

is a generalized symmetry of the augmented system

$$\Xi_a = 0, \qquad \frac{\partial \gamma_b}{\partial x^i} = X_{ib}. \tag{10}$$

In other words, if (G^α, H_b) *is a nonlocal* π*-symmetry of the system* $\Xi_a = 0$*, the linearized equations*

$$\Xi_{a,\tau} = 0 \qquad \text{and} \qquad \frac{\partial}{\partial \tau}\left(\frac{\partial \gamma_b}{\partial x^i}\right) = X_{ib,\tau} \tag{11}$$

with $u^\alpha_\tau = G^\alpha$ *and* $\gamma_{b,\tau} = H_b$*, are satisfied whenever* $u^\alpha(x^i)$ *and* $\gamma_b(x^i)$ *are solutions to (10). Conversely, if (9) is a generalized symmetry of the system (10), then* (G^α, H_b) *is a nonlocal* π*-symmetry of* $\Xi_a = 0$*, in which* $\pi = (\gamma_b, X_{ib}, \widetilde{D}_i)$ *and* $\widetilde{D}_i = D_i + \sum_{b=1}^{N} X_{ib}\partial/\partial\gamma_b$*.*

Proposition 1 is proven in [39], and several applications can be found in [23, 32, 36, 37, 39]. Since generalized symmetries transform solutions into solutions [33], proposition 1 implies the following important result:

Corollary 1. *If* $u^\alpha_0(x^i)$ *and* $\gamma^0_b(x^i)$ *are solutions to the augmented system (10), the solution to the Cauchy problem*

$$\frac{\partial u^\alpha}{\partial \tau} = G^\alpha, \qquad \frac{\partial \gamma_b}{\partial \tau} = H_b, \qquad u^\alpha(x^i, 0) = u^\alpha_0(x^i), \qquad \gamma_b(x^i, 0) = \gamma^0_b(x^i),$$

is a one-parameter family of solutions to the augmented system (10). In particular, nonlocal symmetries send solutions to the original system $\Xi_a = 0$ *to solutions of the same system.*

3. Shallow water equations

This section is on pseudo-potentials and nonlocal symmetries for the equations due to Korteweg and de Vries [7, 8, 27],

$$\alpha u_t = \beta u_{xxx} + \gamma u u_x, \tag{12}$$

Camassa and Holm [11],

$$m = u_{xx} - u, \qquad m_t = -m_x u - 2m u_x, \tag{13}$$

and Hunter and Saxton [24],

$$m = u_{xx}, \qquad m_t = -m_x u - 2m u_x. \tag{14}$$

Henceforth ϵ and ν will denote real parameters. The two-parameter family of equations

$$-2\nu^2 U_X U_{XX} + 3\epsilon U_X U - \nu^2 U_{XXX} U + \tfrac{2}{3}(1-\nu^{5/2})U_{XXX} + \epsilon U_T - \nu^2 U_{XXT} = 0 \tag{15}$$

was introduced in [38], and clearly it includes the KdV, CH and HS equations as special cases. With the choices $\epsilon = 1$ and $1 - \nu^{5/2} = \gamma$, equation (15) has been derived as a shallow water equation by Dullin, Gottwald and Holm [20] via an asymptotic expansion of the Euler equations. Also noteworthy is the fact that (15) can be interpreted as a geodesic equation on the Virasoro group (it is equation (3.9) of [26] if the parameters β, α and b appearing there are replaced by ν^2, ϵ, and $(2/3)(1-\nu^{5/2})$ respectively) and that it possesses a bi-Hamiltonian formulation which can be deduced from [22]; see [38].

Remark 1. On bi-Hamiltonian PDEs and Virasoro group: The H^k Sobolev inner product on the Lie algebra *vir* of the Virasoro group *Vir* is defined as [26, 18]

$$\langle (f,a),(g,b)\rangle_k = \int_{S^1} \sum_{i=0}^{k} \left(\partial_x^i f\right)\left(\partial_x^i g\right) \mathrm{d}x + ab,$$

in which $(f,a),(g,b) \in vir$ and *vir* is identified with $C^\infty(S^1) \times \mathbf{R}$. The KdV equation corresponds to the geodesic flow on the Virasoro group with respect to the right-invariant metric induced by the H^0 inner product, and CH to the geodesic flow on *Vir* with respect to the right-invariant metric induced by the H^1 inner product (see [26] and references therein). One can also define a degenerate form on *vir* by $\langle (f,a),(g,b)\rangle = \int \partial_x f \partial_x g \, dx + ab$ and extend to a degenerate right-invariant metric on *Vir*. The Hunter–Saxton equation describes the geodesic flow on an homogeneous space of *Vir* on which this metric becomes nondegenerate [26]. The geodesic flows on *Vir* obtained with the help of other choices of H^k Sobolev inner products *do not* possess natural bi-Hamiltonian structures [17, 18, 26].

3.1. Pseudopotentials

As is well known, the existence of quadratic pseudo-potentials for a given system of equations $\Xi_a = 0$ follows from the existence of an $sl(2,\mathbf{R})$-valued linear problem associated with $\Xi_a = 0$. Proposition 6 was obtained in [38] as a corollary to the fact that equation (15) is of pseudo-spherical type. It may be checked by direct computations:

Proposition 2. *Equation (15) is the integrability condition of the one-parameter family of linear problems* $d\psi = (X\,dx + T\,dt)\psi$*, in which the matrices X and T are given by*

$$X = \begin{bmatrix} 0 & (1/3)\epsilon\zeta + (1/3)\sqrt{\nu}\epsilon + \nu^2 U_{XX} - \epsilon U \\ (3/4)(1+\zeta\nu^2)^{-1} & 0 \end{bmatrix} \tag{16}$$

and

$$T = \begin{bmatrix} -\frac{1}{2}U_X & \begin{matrix} -\frac{2}{3}\nu^{5/2}U_{XX} + \frac{1}{3}\epsilon U\zeta - \frac{2}{9}\nu\epsilon \\ +\frac{1}{3}\sqrt{\nu}\epsilon U - \frac{4}{9}\sqrt{\nu}\epsilon\zeta - \frac{2}{9}\epsilon\zeta^2 \\ +\frac{2}{3}U_{XX} - \nu^2 U_{XX}U + \epsilon U^2 \end{matrix} \\ -\dfrac{(3/4)U + (1/2)\zeta + (1/2)\sqrt{\nu}}{1+\zeta\nu^2} & \frac{1}{2}U_X \end{bmatrix}, \tag{17}$$

and ζ *is an arbitrary real parameter.*

Example 1. If $\nu = 1$ and $\lambda = (2/3)(1+\zeta)$, equations (16) and (17) yield the matrices

$$X = \frac{1}{2}\begin{bmatrix} 0 & \epsilon\lambda + 2m \\ \lambda^{-1} & 0 \end{bmatrix}, \qquad T = \frac{1}{2}\begin{bmatrix} -U_X & -2Um + \epsilon\lambda U - \epsilon\lambda^2 \\ -1 - U\lambda^{-1} & U_X \end{bmatrix}, \tag{18}$$

in which $m = U_{XX} - \epsilon U$. This is the associated linear problem for the CH and HS equations derived in [36, 37].

Corollary 2. *The equation*

$$-2\nu^2 U_X U_{XX} + 3\epsilon U_X U - \nu^2 U_{XXX}U + \tfrac{2}{3}(1-\nu^{5/2})U_{XXX} + \epsilon U_T - \nu^2 U_{XXT} = 0 \tag{19}$$

admits a quadratic pseudo-potential $\gamma(X,T)$ *determined by the equations*

$$-\gamma_X = \frac{3}{4(1+\zeta\nu^2)}\gamma^2 - \left(\nu^2 U_{XX} - \epsilon U + \frac{1}{3}\epsilon(\zeta + \sqrt{\nu})\right), \tag{20}$$

$$-\gamma_T = \frac{1}{2(1+\zeta\nu^2)}\left(-\frac{3}{2}U - \zeta - \sqrt{\nu}\right)\gamma^2 + U_X\gamma + \left(U(\nu^2 U_{XX} - \epsilon U) + \frac{2}{3}(\nu^{5/2} - 1)U_{XX} - \frac{1}{3}\epsilon(\zeta + \sqrt{\nu})U + \frac{2}{9}\epsilon(\zeta^2 + 2\sqrt{\nu}\zeta + \nu)\right) \tag{21}$$

in which ζ *is an arbitrary real parameter.*

Example 2. Taking $\nu = 0$ and $\epsilon = 1$ in (20) gives $-\gamma_X = (3/4)\gamma^2 + U - (1/3)\zeta$, and therefore one recovers the usual Miura transformation for the KdV equation. On the other hand, if one sets $\nu = 1$ and $\lambda = (2/3)(1+\zeta)$ in (20) and (21), one obtains

$$U_{XX} - \epsilon U = \gamma_X + \frac{\gamma^2}{2\lambda} - \frac{\epsilon}{2}\lambda, \tag{22}$$

$$-\gamma_T = -\frac{1}{2}\left(\frac{U}{\lambda} + 1\right)\gamma^2 + U_X\gamma + \left(U(U_{XX} - \epsilon U) - \frac{1}{2}\epsilon U\lambda + \frac{1}{2}\epsilon\lambda^2\right). \tag{23}$$

Substitution of (22) into (23) implies that the Camassa–Holm equation (13) and the Hunter–Saxton equation (14) possess the parameter-dependent conservation law

$$\gamma_T = \lambda\left(U_X - \gamma - \frac{1}{\lambda}U\gamma\right)_X. \tag{24}$$

As in the KdV case, one can use (22) and (24) to construct sequences of conservation laws for the CH and HS equations [11, 21, 25, 30, 36, 37]. It is therefore natural to postulate equation (22) as the analogue of the Miura transformation for the CH and HS equations, and (24) as the corresponding modified equation [36, 38].

3.2. Nonlocal symmetries

Substitution of (20) into (21) yields the conservation law

$$\gamma_T = \left[\tfrac{2}{3}(\zeta\nu^2 + 1)U_X - \tfrac{2}{3}(\zeta + \sqrt{\nu})\gamma - \gamma U\right]_X. \tag{25}$$

In analogy with the Camassa–Holm case [36], this conservation law allows one to obtain the following result:

Theorem 1. *Set $m = \nu^2 U_{XX} - \epsilon U$, and let γ and δ be defined by the equations*

$$\begin{aligned} \gamma_X &= \frac{-3}{4(1+\zeta\nu^2)}\gamma^2 + \left(m + \frac{1}{3}\epsilon(\zeta + \sqrt{\nu})\right), \\ \gamma_T &= \left[\frac{2}{3}(\zeta\nu^2 + 1)U_X - \frac{2}{3}(\zeta + \sqrt{\nu})\gamma - \gamma U\right]_X, \end{aligned} \tag{26}$$

and

$$\delta_X = \gamma, \qquad \delta_T = \tfrac{2}{3}(\zeta\nu^2 + 1)U_X - \tfrac{2}{3}(\zeta + \sqrt{\nu})\gamma - U\gamma, \tag{27}$$

which are compatible on solutions of (19). The function

$$G = \gamma \exp\left((3/2)\frac{\delta}{1+\zeta\nu^2}\right) \tag{28}$$

is the shadow of a nonlocal symmetry for equation (19).

Now one extends the shadow (28) to a bonafide nonlocal symmetry. Intuitively, as anticipated in [23, 31, 32] and explained in [39], in order to make this extension one needs to find the infinitesimal variations of the functions γ and δ as U is infinitesimally deformed into $U \mapsto U + \tau G$, in which G is given by (28). Once this is done, proposition 1 allows one to conclude that a genuine nonlocal symmetry has been found.

Theorem 2. *Write equation (19) as a system of equations for two variables, m and U, as*

$$m = \nu^2 U_{XX} - \epsilon U, \qquad m_T = -m_X U - 2mU_X + \tfrac{2}{3}(1 - \nu^{5/2})U_{XXX}, \tag{29}$$

and let γ, δ and β be defined by the equations

$$\gamma_X = -\frac{3}{4(1+\zeta\nu^2)}\gamma^2 + \left(m + \frac{1}{3}\epsilon(\zeta + \sqrt{\nu})\right), \tag{30}$$

$$\gamma_T = \left[\frac{2}{3}(\zeta\nu^2+1)U_X - \frac{2}{3}(\zeta+\sqrt{\nu})\gamma - \gamma U\right]_X, \tag{31}$$

$$\delta_X = \gamma, \tag{32}$$

$$\delta_T = \frac{2}{3}(\zeta\nu^2+1)U_X - \frac{2}{3}(\zeta+\sqrt{\nu})\gamma - U\gamma, \tag{33}$$

$$\beta_X = \left[\nu^2 m + \frac{1}{3}\epsilon(\nu^{5/2}-1)\right]\exp\left((3/2)\frac{\delta}{1+\zeta\nu^2}\right), \tag{34}$$

$$\beta_T = \left[-\frac{1}{3}(\nu^{5/2}-1)(2m+\epsilon U) - \frac{1}{2}\gamma^2 + \frac{2}{9}\epsilon(2\zeta + \zeta^2\nu^2 - \nu^3 + 2\sqrt{\nu}) - \nu^2 U m\right] \times \exp\left((3/2)\frac{\delta}{1+\zeta\nu^2}\right), \tag{35}$$

which are compatible on solutions of (29). The system of equations (29)–(35) possesses the symmetry

$$W = \gamma\exp\left((3/2)\frac{\delta}{1+\zeta\nu^2}\right)\frac{\partial}{\partial U} + \left[\nu^2 m_X + \frac{3\nu^2\gamma}{1+\zeta\nu^2}m + \gamma\epsilon\frac{\nu^{5/2}-1}{1+\zeta\nu^2}\right] \times\exp\left((3/2)\frac{\delta}{1+\zeta\nu^2}\right)\frac{\partial}{\partial m} + \left[\nu^2 m + \frac{1}{3}\epsilon(\nu^{5/2}-1)\right]\exp\left((3/2)\frac{\delta}{1+\zeta\nu^2}\right)\frac{\partial}{\partial\gamma} + \beta\frac{\partial}{\partial\delta} + \left(\nu^2\left[\nu^2 m + \frac{1}{3}\epsilon(\nu^{5/2}-1)\right]\exp\left(3\frac{\delta}{1+\zeta\nu^2}\right) + \frac{3}{4(1+\zeta\nu^2)}\beta^2\right)\frac{\partial}{\partial\beta}. \tag{36}$$

Note that W *is not* the evolutionary representative of a classical symmetry of (29)–(35), as it can be seen by using [33, Chapter 5]. The vector field W is a genuine first-order generalized symmetry for the system of equations (29)–(35). Proposition 1 implies the following corollary.

Corollary 3. *The vector field (36) determines a nonlocal symmetry of the system of equations (29).*

One can find the flow of the vector field (36) by integrating a system of first-order partial differential equations. The system one needs to consider is

$$\frac{\partial U}{\partial\tau} = \gamma\exp\left((3/2)\frac{\delta}{1+\zeta\nu^2}\right) \tag{37}$$

$$\frac{\partial m}{\partial\tau} = \left[\nu^2 m_X + \frac{3\nu^2\gamma}{1+\zeta\nu^2}m + \gamma\epsilon\frac{\nu^{5/2}-1}{1+\zeta\nu^2}\right]\exp\left((3/2)\frac{\delta}{1+\zeta\nu^2}\right), \tag{38}$$

$$\frac{\partial\gamma}{\partial\tau} = \left[\nu^2 m + \frac{1}{3}\epsilon(\nu^{5/2}-1)\right]\exp\left((3/2)\frac{\delta}{1+\zeta\nu^2}\right), \tag{39}$$

$$\frac{\partial\delta}{\partial\tau} = \beta, \tag{40}$$

$$\frac{\partial\beta}{\partial\tau} = \nu^2\left[\nu^2 m + \frac{1}{3}\epsilon(\nu^{5/2}-1)\right]\exp\left(3\frac{\delta}{1+\zeta\nu^2}\right) + \frac{3}{4(1+\zeta\nu^2)}\beta^2, \tag{41}$$

with initial conditions $U(X,T,0) = U_0$, $\gamma(X,T,0) = \gamma_0$, $\delta(X,T,0) = \delta_0$ and $\beta(X,T,0) = \beta_0$ in which $U_0(X,T)$, $\gamma_0(x,T)$, $\delta_0(X,T)$ and $\beta_0(X,T)$ are particular solutions to (29)–(35). General theorems on existence, uniqueness and regularity of solutions to symmetric hyperbolic systems such as (37)–(41) are known (see for instance [41]), and therefore one can conclude that the foregoing construction determines two-parameter families of solutions to equation (29). From a computational point of view, one can simplify the analysis of (37)–(41) by noticing that equations (38)–(41) do not depend on U, and can therefore be solved separately. Full details appear in [38].

3.3. Remarks on nonlocal symmetries and integro-differential equations

While one can eliminate the variable m from (29)—and therefore write (29) as a partial differential equation for U—if one considers (29) as an equation for m only, this system becomes an integro-differential equation. In the Camassa–Holm case, this integro-differential approach has proven to be crucial: the properties of m determine whether solutions to CH are global in time or represent breaking waves [13], and the nonlocal operator $\left(1 - D_x^2\right)^{-1}$ appears prominently in a recent construction of conservation laws [30], in the study of peakon dynamics and weak solutions [11, 14], in the scattering/inverse scattering approach to CH [3, 19], and also in the rigorous proof that the least action principle holds for CH [16].

In the context of symmetry analysis of the general equation (29), the inverse of the operator $\epsilon - \nu D_x^2$ is also important in the following sense.

Theorem 3. *If the variables m and U are related by*

$$m = \nu^2 U_{XX} - \epsilon U, \qquad m_T = -m_X U - 2mU_X + \tfrac{2}{3}(1-\nu^{5/2})U_{XXX}, \tag{42}$$

the functions γ, δ and β are defined by means of equations (30)–(35), and m, γ, δ, β satisfy equations (38)–(41), then U satisfies equation (37), namely,

$$\frac{\partial U}{\partial \tau} = \gamma \exp\left((3/2)\frac{\delta}{1+\zeta\nu^2}\right). \tag{43}$$

The proof is straightforward: compute the derivative $\beta_{T,\tau}$ using (35), compute $\beta_{\tau,T}$ using (41), and then simplify both calculations using (30)–(35) and (38)–(41). In the Camassa–Holm equation case, this result is intuitively clear from an analytic point of view, since the operator $1 - D_x^2$ connecting U and m is an isomorphism between Sobolev spaces (see for example [14, 30]). From the point of view of formal geometry, however, one would like to interpret this result as stating that the integro-differential equation

$$m_T = -m_X U - 2mU_X + \tfrac{2}{3}(1-\nu^{5/2})U_{XXX},$$

in which $U = (\nu^2 D_{XX} - \epsilon)^{-1} m$, possesses the *nonlocal* symmetry

$$\begin{aligned} V = {} & \left[\nu^2 m_X + \frac{3\nu^2\gamma}{1+\zeta\nu^2} m + \gamma\epsilon\frac{\nu^{5/2}-1}{1+\zeta\nu^2}\right] \exp\left((3/2)\frac{\delta}{1+\zeta\nu^2}\right)\frac{\partial}{\partial m} \\ & + \left[\nu^2 m + \frac{1}{3}\epsilon(\nu^{5/2}-1)\right]\exp\left((3/2)\frac{\delta}{1+\zeta\nu^2}\right)\frac{\partial}{\partial\gamma} \\ & + \beta\frac{\partial}{\partial\delta} + \left(\nu^2\left[\nu^2 m + \frac{1}{3}\epsilon(\nu^{5/2}-1)\right]\exp\left(3\frac{\delta}{1+\zeta\nu^2}\right) + \frac{3}{4(1+\zeta\nu^2)}\beta^2\right)\frac{\partial}{\partial\beta}, \end{aligned}$$

where the nonlocal variables γ, δ, β are determined by (30)–(35). It appears that this kind of symmetries has not been studied in the literature: versions of a symmetry theory for integro-differential equations are for instance in [12, 29, 44], but it would seem that they do not cover the present example.

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